Euler potentials for two layers with non-constant current densities in the ambient magnetic field aligned to the layers

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Abstract. The Euler potentials for two current layers aligned to an ambient homogeneous magnetic field are found. Previous treatment of such a system assumed constant current density in the layers. However, the magnetic field becomes infinite at the edges. The new approach eliminates this inconsistency by introducing an inhomogeneous current density. Euler potentials are constructed semi-analytically for such a system. Charged-particle motion and trapping in it are examined by this representation. Using Euler potentials, the influence of current sheets of zero and non-zero thicknesses on energetic-particle fluxes is investigated, and characteristic flux variations near the sheets are presented. The results can be applied to Birkeland currents.

Keywords. Magnetospheric physics (current systems)

1 Introduction

Euler potentials represent an alternative way of describing a magnetic field. They are two scalar functions, usually denoted as \(\alpha\) and \(\beta\), related to the magnetic field vector by

\[
B = \nabla\alpha \times \nabla\beta.
\]  

(1)

From the definition (Eq. 1) it is clear that Euler potentials are not unique, but they are constant along magnetic field lines. This property makes them a convenient tool for studies of charged-particle motions in related magnetic fields (Gardner, 1959; Northrop and Teller, 1960; Ray, 1963). In recent years they are applied to a description of magnetic fields and charged-particle dynamics in Earth’s magnetosphere-ionosphere (Schulz and Chen, 2008; Zaharia, 2008; Gkioulidou et al., 2012; Wang et al., 2014).

Romashets and Vandas (2012) have analytically obtained Euler potentials for a system where electric currents in two sheets flow along background magnetic field lines, that is, for situations relevant to Birkeland currents in the auroral magnetosphere-ionosphere. Birkeland currents are currents flowing along magnetic field lines in two sheets (Schield et al., 1969; Vasylunias, 1970; Russell and Fleishman, 2002). Originally deduced from ground-based magnetic field variation measurements during auroral phenomena, they were confirmed by in situ measurements in the 1970s (Zmuda and Armstrong, 1974; Iijima and Potemra, 1976a, b). Current observations enable the construction of maps of Birkeland currents in detail (e.g., Friis-Christensen and Lassen, 1991; Russell et al., 1997; Waters et al., 2001; Korth et al., 2010; Anderson et al., 2014). The model by Romashets and Vandas (2012) was simple: current sheets had zero thickness and uniform current distribution inside them. This system has a feature which is unnatural; that is, the magnetic field diverges (becomes infinite) at the sheet edges. In order to fix this unwanted property, Vandas and Romashets (2014) introduced a non-zero thickness of the sheets. However, for small thicknesses of the sheets (which are relevant to Birkeland currents) the magnetic field at the edges becomes too high because this system resembles a zero-thickness sheet configu-
ration. In this paper we propose an alternative way of fixing
the field divergence: the sheets will retain their zero thick-
ness, but the distribution of the current inside them will not
be uniform. Section 2 describes derivation of Euler potentials
for two current sheets aligned with a uniform ambient mag-
netic field. In Sect. 3 new results are compared to the case
of sheets with uniform currents and similarities/differences
are outlined. Section 4 uses derived Euler potentials to in-
vestigate the influence of current sheets on energetic-particle
fluxes.

2 Euler potentials

Figure 1a shows the considered current system. Two sheets
with varying surface current density are situated parallel to
the z axis, they are boundless along it, they are separated by
$2y_0$ in the x direction, and extend by $2y_0$ in the y direction.
The currents in the sheets can be parallel or anti-parallel to
each other: denoting the surface current density at the right
sheet by $j$, the left sheet will have $\pm j$ for the same y (the
“±” nomenclature will be used throughout this section and
means parallel, “+”, or anti-parallel, “−”, cases). The sur-
face current density in the right sheet is constant and equal
to $j_0 > 0$ within $|y| < y_1 (0 < y_1 < y_0)$ and then linearly de-
creases to zero towards the edges ($y = y_0$ or $y = −y_0$), so its
profile is trapezoidal (see Fig. 2).

First we consider one sheet only and centered at the origin.
The magnetic field is determined from the Biot–Savart law:

$$B(r) = \frac{\mu_0}{4\pi} \int \frac{j(r') \times (r - r')}{|r - r'|^3} \, \mathrm{d}S.$$  \hfill (2)

In our case we have $j = j(y)e_y$, where $e_y$ is the unit vector
in the direction of the z axis, so

$$B^{(1)}(x, y, z) = \frac{\mu_0}{2\pi} \int_{-y_0}^{y_0} \frac{[(y' - y)j(y'), yj(y'), 0]}{x^2 + (y' - y)^2} \, \mathrm{d}y',$$  \hfill (3)

where the integration over $y'$ has already been performed.
Assuming the trapezoidal surface current profile (Fig. 2), the
resulting field becomes

$$B^{(1)}_x = B_i \frac{x_0}{y_0 + y_1} \left[ \frac{x \left( \arctan \frac{y + y_1}{x} + \arctan \frac{y - y_1}{x} \right)}{y_0 + y_1} - \frac{y + y_0}{y - y_0} \right] + \frac{1}{2} \ln \frac{x^2 + (y - y_1)^2}{x^2 + (y + y_1)^2} - \frac{y + y_0}{2(y_0 - y_1)} \ln \frac{x^2 + (y - y_0)^2}{x^2 + (y + y_0)^2},$$  \hfill (4)

$$B^{(1)}_y = \frac{x_0}{y_0 + y_1} \left[ \frac{\arctan \frac{y + y_1}{x} - \arctan \frac{y - y_1}{x}}{y_0 + y_1} \right] + \frac{y + y_0}{y_0 - y_1} \left( \arctan \frac{y + y_0}{x} - \arctan \frac{y + y_1}{x} \right) + \frac{y - y_0}{y_0 - y_1} \left( \arctan \frac{y - y_0}{x} - \arctan \frac{y - y_1}{x} \right) - \frac{x}{2(y_0 - y_1)} \ln \frac{\left( x^2 + (y + y_0)^2 \right)^{y_0 - y_1} \left( x^2 + (y + y_1)^2 \right)^{y_0 - y_1}}{\left( x^2 + (y + y_1)^2 \right)^{y_0 - y_1} \left( x^2 + (y + y_0)^2 \right)^{y_0 - y_1}},$$  \hfill (5)

where $B_i$ is introduced via the total current $I_0$ of the sheet,
$B_i = \mu_0 I_0 / (2\pi x_0)$ and $I_0 = j_0(y_0 + y_1)$ ($\mu_0$ is the magnetic
vacuum permeability).

One can verify that the boundary conditions for the mag-
netic field at the current sheet are satisfied. They generally
read $n \cdot (B_2 - B_1) = 0$ and $n \times (B_2 - B_1) = \mu_0 j$, where $n$
is a unit normal vector to the boundary pointing from region 1

![Figure 1](image-url)
to region 2, and \( j \) is the surface current flowing on the boundary. In our case if region 1 has \( x < 0 \) and \( |y| < y_0 \), we have \( n = (1, 0, 0) \), \( B_1 = \lim_{x \to 0^+} B^{(1)} \), and \( B_2 = \lim_{x \to 0^-} B^{(1)} \). Direct calculations using Eqs. (4)–(6) prove that the boundary conditions are met.

Magnetic field at the edge is

\[
B^{(1)}_{\text{edge}} = \lim_{y \to y_0+} \lim_{x \to 0^-} B^{(1)}(x, y, z) = B_1 \left( \frac{y_0 - y_1}{y_0 + y_1} \ln \frac{2y_0 + y_1}{2y_0 - y_1} - 1 \right) - 2 \frac{y_0 - y_1}{y_0 + y_1} \ln \frac{2y_0 + y_1}{2y_0 - y_1} 0, 0, 0. \tag{7}
\]

It is limited but for case of the uniform current density

\[
B^{(1)}_{\text{edge}} = \lim_{y \to y_0^{-}} \lim_{x \to \infty} B^{(1)}(x, y, z) = (\infty, 0, 0). \tag{8}
\]

Magnetic field of our system is given by

\[
B^\pm(x, y, z) = B^{(1)}(x + x_0, y, z) \pm B^{(1)}(x - x_0, y, z) + B_0 \tag{9}
\]

with the ambient homogeneous field \( B_0 = (0, 0, B_0) \). It follows from the fact that the Biot–Savart law is linear in currents, so the field from more currents is simply a superposition of the fields generated by the particular currents. We have two currents in the system, and the integral (Eq. 2) splits into two integrals, the first one for the left sheet with \( x' = -x_0 \) and the second one for the right sheet with \( x' = x_0 \). The boundary conditions at the sheets are met. Let us consider the right sheet. The magnetic field generated by the current here fulfills the boundary conditions as it has been already shown. The field from the left sheet and the homogeneous field \( B_0 \) are continuous here, so they give no contributions to jumps.

When \( y_1 \to y_0 \), the field (Eq. 9) tends to the uniform current case (Romashets and Vandas, 2012). When \( y_1 \to 0 \), the current has a triangular profile. Figure 1a and c show magnetic field magnitude distribution without the ambient field \( B_0 \) when \( y_0 = x_0 \) and \( y_1 = 0.9y_0 \) for the parallel and antiparallel cases, respectively. The thick vertical lines depict two current sheets stretching from \( y = -y_0 \) to \( y_0 \) at \( x = -x_0 \) and \( x = x_0 \), respectively. The magnetic field magnitude remains limited at the edges of the sheets (\( y = y_0 \) or \( y = -y_0 \)).

First we search Euler potentials \( \alpha^{(1)} \) and \( \beta^{(1)} \) for one current sheet centered at the origin and without the external field \( (B_0 = 0) \), that is, for the field given by Eqs. (4)–(6). We set

\[
\alpha^{(1)} = B_1 x_0 y \tag{10}
\]

and from the definition of Euler potentials we have

\[
\begin{align*}
B^{(1)} &= \text{grad} \alpha^{(1)} \times \text{grad} \beta^{(1)} \\
&= B_1 x_0 \left( -\frac{\partial \beta^{(1)}}{\partial y}, \frac{\partial \beta^{(1)}}{\partial x}, 0 \right), \tag{11}
\end{align*}
\]

so we can calculate

\[
\begin{align*}
\beta^{(1)} &= -\frac{1}{B_1 x_0} \int B^{(1)} \, dy \\
&= \frac{1}{4(y_0^2 - y_1^2)} \left[ 4x \left( (y + y_0) \arctan \frac{y + y_0}{x} - (y + y_1) \arctan \frac{y + y_1}{x} \right) - (x^2 - y_1^2) (x^2 + y^2) \right. \\
&\left. + 2y \left( y_0 \ln \frac{x^2 + (y + y_0)^2}{x^2 + (y - y_0)^2} - y_1 \ln \frac{x^2 + (y + y_1)^2}{x^2 + (y - y_1)^2} \right) \right] \\
&\left. + y_0^2 \ln \frac{x^2 + (y + y_0)^2}{x^2 + (y - y_0)^2} \right] - \frac{1}{2} \tag{12}
\end{align*}
\]

The \( x_0 \) in denominators here has only the function to keep arguments dimensionless and to ensure that, in the limit \( y_1 \to y_0 \), the potential \( \beta^{(1)} \) tends to that in Romashets and Vandas (2012).

We construct a set of curvilinear orthogonal coordinates \( \mu^{(1)} \) and \( \eta^{(1)} \). It is supposed that

\[
\begin{align*}
\frac{\partial \mu^{(1)}}{\partial x} &= -\frac{\partial \eta^{(1)}}{\partial y}, \quad \frac{\partial \mu^{(1)}}{\partial y} = \frac{\partial \eta^{(1)}}{\partial x}, \tag{13}
\end{align*}
\]

and set \( \mu^{(1)} = \beta^{(1)} \). The coordinates will be orthogonal because it is a vacuum field (cf. Vandas and Romashets, 2014, Appendix A). We have from Eqs. (13) and (11)

\[
\begin{align*}
\nu^{(1)} &= \int \frac{\partial \mu^{(1)}}{\partial y} \, dx + \int \frac{\partial \beta^{(1)}}{\partial y} \, dx = -\frac{1}{B_1 x_0} \int B^{(1)} \, dx \\
&= \frac{1}{2(y_0^2 - y_1^2)} \left[ (x^2 - (y + y_0)^2) \arctan \frac{y + y_0}{x} + x \left( y_0 \ln \frac{x^2 + (y + y_0)^2}{x^2 + (y - y_0)^2} + (y - y_1) \ln \frac{x^2 + (y + y_1)^2}{x^2 + (y - y_1)^2} \right) \right]. \tag{14}
\end{align*}
\]

The curvilinear coordinates of the system of two current sheets (plus \( B_0 \)) are

\[
\mu^{(1)} = \mu^{(1)}(x + x_0, y) \pm \mu^{(1)}(x - x_0, y) \tag{15}
\]
\[
\eta^+(x, y) = \eta^{(1)}(x + x_0, y) \pm \eta^{(1)}(x - x_0, y) + k^\pm \pi, \\
k^+ = 0 \quad |x| < x_0, \\
= \text{sign } xy \quad |x| > x_0, \\
k^- = \text{sign } y \quad |x| < x_0, \\
= 0 \quad |x| > x_0,
\]

where sign is the signum function. In the limit \(y_1 \to y_0\) (uniform current case) the coordinates \(\mu^\pm\) and \(\eta^\pm\) transform into the coordinates by the same symbols in Romashets and Vandas (2012). It holds that

\[
\mu^+(x, y), \mu^-(x, y), \eta^+(x, y), \eta^-(x, y) = -\eta^-(x, y),
\]

There is a discontinuity at the \(x\) axis in \(\eta^\pm\):

\[
\eta^+(x, 0+) - \eta^-(x, 0-) = 2\pi \text{ sign } x \quad |x| > x_0, \\
\eta^-(x, 0+) - \eta^+(x, 0-) = 2\pi \text{ for } |x| < x_0.
\]

Regarding the coordinate \(\eta^\pm\) as a cyclic variable, this discontinuity is removed. However, \(\eta^\pm\) remains discontinuous at the current sheets. The coordinate \(\mu^\pm\) is continuous everywhere.

Figure 1b and d show the coordinates \(\mu^\pm\) (thick red lines) and \(\eta^\pm\) (thin blue lines). These figures are very similar to corresponding figures of the uniform-current case (Romashets and Vandas, 2012), so labeling is nearly the same. Both coordinates are displayed with step of 0.25. The coordinate \(\mu^+\) (Fig. 1b) is labeled vertically in the belt between \(\eta^+ = -0.5\) and \(-0.25\) from 2 to 4; an additional value of 2 is in the belt between \(\eta^+ = 0.25\) and 0.5. The coordinate \(\eta^+\) is labeled clockwise in the belt between \(\mu^+ = 3.25\) and 4 from \(-3\) to 3. The \(y\) axis has \(\mu^+ = 0\), and the \(x\) axis has \(\eta^+ = \pm\pi\) for \(|x| > x_0\) (the places of discontinuity are plotted by the thin blue dashed lines) and value 0 between \(-x_0\) and \(x_0\) (see additional digit 0 there). The \(\mu^-\) (Fig. 1d) is labeled clockwise in the belt between \(\eta^- = 0.5\) and 0.75 from \(-1\) to 1. The \(y\) axis has \(\mu^- = 0\). The \(\eta^-\) is labeled vertically in the belt between \(\mu^- = -0.5\) and \(-0.25\) (with exception of the value 0 at the \(x\) axis). The \(x\) axis has \(\eta^- = 0\) for \(|x| > x_0\) and \(\pm\pi\) between \(-x_0\) and \(x_0\) (again, the place of discontinuity is plotted by the thin blue dashed line). One can see that there are only minor differences between Fig. 1 and its limiting case \(y_1 = y_0\) (Fig. 1 of Romashets and Vandas, 2012).

To construct the pair \(\alpha^\pm, \beta^\pm\) we set

\[
\alpha^\pm = B_x x_0 \zeta + B_0 x_0^2 f^\pm(\mu^\pm, \eta^\pm), \\
\beta^\pm = \mu^\pm,
\]

where

\[
f^\pm = \frac{1}{x_0} \int_0^{\eta^\pm} \left[ \left( \frac{\partial \mu^\pm}{\partial x} \right)^2 + \left( \frac{\partial \mu^\pm}{\partial y} \right)^2 \right]^{-1} \, d\eta^\pm
\]

(cf. Romashets and Vandas, 2012). The integration is for a constant \(\mu^\pm\), i.e., along a magnetic field line (\(\eta^\pm\) parameterizes the line). The thick red lines in Fig. 1b and d also are projections of magnetic field lines which are helical in the \(z\) direction and lie on magnetic surfaces defined by \(\mu^\pm = \text{const}\).

To check our derivation, we compared these lines with magnetic field lines following from the direct numerical integration using the field (Eq. 9). They were coincident.

### 3 Comparison to sheets with uniformly distributed currents

We shall consider anti-parallel case, which has relevance to Birkeland currents, and drop “−” from quantities for clarity. For calculations we took the same values of constants in the model as used in Romashets and Vandas (2012): \(B_0 = 60000\) nT, \(B_1 = 60000\) nT, and \(x_0 = 150\) km. The relatively high value of \(B_1\) represents a hypothetic geomagnetic superstorm.

Figure 3 shows lead \(\Delta z\) of helical magnetic field lines as a function of \(y_1\) for three sheet widths \(y_0\) (\(\Delta z\) is the length along the \(z\) axis over which the field line makes a full rotation). Solid lines are labeled by their \(y_0\) values and their thickness increases with increasing \(y_0\). All profiles are for one particular magnetic field line, that is, the line which crosses the edge of the sheet. One can see that when \(y_0\) increases, the field lines become less helical. For a given \(y_0\) the lead \(\Delta z\) decreases with \(y_1\), but for lower \(y_0\) it is nearly constant. The values of \(\Delta z\) at the limit \(y_1 = 0\) coincide with places of bullets which can be found in Fig. 2 of Romashets and Vandas (2012); \(\mu\) slightly changes along each profile, increases with rising \(y_1\), approximately from 1.53 to 1.73 for \(y_0 = x_0/2\), from 0.93 to 1.13 for \(y_0 = x_0\), and from 0.46 to 0.66 for \(y_0 = 2x_0\).
Examples of particle mirroring at magnetic field lines for \( y_0 = x_0/2 \) are shown in Fig. 4. Figure 4a is taken from Romashets and Vandas (2012) (Fig. 3b there) and represents the limiting case \( y_1 = y_0 \). Figure 4b is made in the same format, but for \( y_1 = 0.9y_0 \). Magnetic field magnitude profiles along three magnetic field lines are plotted by lines with different thicknesses. The dashed lines show mirroring condition (level of magnetic field magnitude at which particles mirror) for particles with initial pitch angle 80\(^\circ\) (the angle between the particle velocity and the magnetic field vector), the particles start from the \( x \) axis. The line thickness corresponds to associated magnetic field lines. The thick line is for a magnetic field line with \( \cosh \mu = 3 \) (\( \mu = 1.763 \)). It crosses the sheet, which is manifested as a jump in \( B \) and \( \eta \). Particles mirror in both regions, \( x > x_0 \) and \( x < x_0 \), and there are two associated dashed lines (with different levels of mirror \( B \)). A magnetic field line with \( \cosh \mu = 2.5 \) (\( \mu = 1.567 \)) is plotted by the line with normal thickness. It does not cross the sheet but has a local \( B \) maximum near \( x = x_0 \). Our particles mirror only in the region \( x > x_0 \). The thin line represents a more distant magnetic field line with \( \cosh \mu = 2 \) (\( \mu = 1.317 \)). It has only one \( B \) maximum (at the \( x \) axis). Here our particles do not mirror. Comparison of Fig. 4a and b indicates that they are quite similar. These figures are supplemented by Fig. 4c and d, showing magnetic-field-line profiles for lines crossing an edge of a sheet. Figure 4c displays the limiting case \( y_1 = y_0 \) as Fig. 4a above it (\( \mu = 1.730 \), cf. description of Fig. 3), while Fig. 4d is for \( y_1 = 0.9y_0 \) like Fig. 4b (\( \mu = 1.665 \)). The magnetic field profiles are very different. The magnitude in Fig. 4c rises to infinity, contrary to Fig. 4d, where its maximum is even lower than in Fig. 4b. Nevertheless conditions for mirroring are quite similar. For slightly higher pitch angles the particles would not mirror in the present case (Fig. 4d) while in the former they do (Fig. 4c). So only a tiny fraction of particles which reach a very vicinity of the edges will mirror differently. Therefore results and conclusions of Romashets and Vandas (2012) remain valid.

Figure 5 displays magnetic field magnitude at the sheet edges as a function of \( y_1 \) for three sheet widths \( y_0 \). As in Fig. 3, the lines are labeled by their \( y_0 \) values and their width indicates the value of \( y_0 \). Magnetic field value remains nearly constant for a wide range of \( y_1 \) values. For \( y_0 \) comparable to \( x_0 \) and larger the field magnitude starts to rise to infinity when \( y_1 \) is very close to \( y_0 \).

Figure 1e shows the situation with magnetic field magnitude near a sheet edge in more detail. It is a magnified part of Fig. 1c around the sheet edge described by the point \([x_0, y_0]\). Magnetic field maximum is not situated at the edge but shifted towards the point where the current density begins to drop, that is, the place with coordinates \([x_0, y_1]\) and stressed by the bullet in the figure.
The current density along the sheet (Fig. 2) is not smooth. Using a higher-order polynomial, we can make it smooth. For example,

\[
j_z = \frac{j_0}{(y_0 - y_1)^3} \left[ 2|y|^3 - 3(y_0 + y_1)y^2 + 6y_0y_1|y| + y_0^2(y_0 - 3y_1) \right]
\]

for \(y_1 < |y| < y_0\), the profile of which is shown in Fig. 6 by the thick line. It is possible to go through the procedures described above and to derive Euler potentials for such a system, but the calculations would be more complex.

4 Hypothetic observations of magnetic field and particle flux variations

In order to compare effects on charged particles of the current systems with zero and non-zero sheet thicknesses, we investigate particle flux variations which would observe a hypothetic spacecraft moving perpendicularly through the sheets. Besides the comparison, the simulated observations may serve as examples of what a spacecraft could measure during a superstorm. At present, such observations are not available because no superstorm with a strength comparable to the 1859 Carrington event (Tsurutani et al., 2003; Siscoe et al., 2006) occurred during the cosmic era.

Consider charged particles moving adiabatically in a magnetic field system. The number density of the particles is given by

\[
n = \int_0^\infty \int_0^{2\pi} f(v, \alpha, \varphi) v^2 \sin \alpha \, d\varphi \, d\alpha \, dv,
\]

where \(f\) is the particle distribution function, \(v\) is the velocity magnitude, \(\alpha\) is the pitch angle, and \(\varphi\) is the phase. A hemispheric flux of these particles along the magnetic field vector reads

\[
F = \int_0^{\pi/2} \int_0^{2\pi} f(v, \alpha, \varphi) v^3 \sin \alpha \cos \alpha \, d\varphi \, d\alpha \, dv.
\]

Let us assume that the initial distribution function is a monoenergetic gyrotropic distribution with a simple anisotropy

\[
f_0(v, \alpha, \varphi) = \frac{n_0}{4\pi v_0^2} (1 + A_1 \cos \alpha) \delta(v - v_0),
\]

where \(n_0\) is the particle number density, \(v_0\) is the particle velocity, \(A_1\) is a coefficient determining anisotropy, and \(\delta\) is the delta function. The hemispheric flux for this distribution function is given by

\[
F_0 = \frac{3 + 2A_1}{12} n_0 v_0.
\]

Figure 7 schematically shows a magnetic field line. It is drawn by a straight line only for simplicity, and it may be curved. The direction of the field vector \(B\) is indicated by the left arrow. The right bullet locates the global magnetic field maximum along the given field line with its value \(B_m\) and the particle flux \(F_m\). A general point where a hypothetic spacecraft observes is drawn by the left bullet and labeled by \(B_1\) and \(F_1\). At this point, the meaning of the hemispheric flux is sketched by a semicircle. The background field is far right, and the flux \(F_1\) is

\[
F_1 = \int_0^{\arcsin(\sqrt{B_1/B_m})} \int_0^{2\pi} f_0(v, \arcsin(\sqrt{B_0/B_m} \sin \alpha), \alpha) v^3 \sin \alpha \cos \alpha \, d\varphi \, d\alpha \, dv
\]

with the Liouville theorem. The flux \(F_m\) is

\[
F_m = \int_0^{\arcsin(\sqrt{B_0/B_m})} \int_0^{2\pi} f_0(v, \arcsin(\sqrt{B_0/B_m} \sin \alpha), \alpha) v^3 \sin \alpha \cos \alpha \, d\varphi \, d\alpha \, dv
\]
\[ F_1/F_0 = B_1/B_m, \]  

so we get

\[ F_1/F_0 = \frac{3b_{0m} + 2A_1[1 - (1 - b_{0m})^2]}{b_{01}(3 + 2A_1)}, \]  

where \( b_{0m} = B_0/B_m \) and \( b_{01} = B_0/B_1 \). Equations (26)–(27) were derived under the assumption that the observation point is behind the field maximum (as it is drawn in Fig. 7). Because our system is periodic, this condition is always fulfilled. However, strictly speaking, due to this periodicity, the asymptotic value \( B_0 \) is not reached like it is indicated by the right arrow. But in a real situation, the sheets are limited in \( z \) and far from them (in the direction of the right arrow) there is the asymptotic value \( B_0 \).

For the initially isotropic distribution \((A_1 = 0)\) we have

\[ F_1/F_0 = \frac{B_1}{B_m}. \]  

Figure 8 shows magnetic field and particle flux variations when a spacecraft moves along the \( x \) axis: it crosses the sheets perpendicularly and in the middle. There are six panels in the figure, and each panel consists of three graphs, showing the relative values of (top) magnetic field magnitude \( B_1 \), (middle) field component generated by the sheets, \( B_y \), and (bottom) hemispheric particle flux \( F_1 \) with respect to the background values (with subscripts 0). There are two lines in the bottom graphs: the thicker line is for an initially isotropic distribution, the thinner one for an anisotropic distribution. The vertical dashed lines are positions of the sheets or their edges (for non-zero thickness). Figure 8a is for the system of sheets with a non-uniform current density and zero thickness (model described in this paper). The sheets have parameters \( y_0/x_0 = 1/2 \) and \( y_1/y_0 = 0.9 \). The presence of the sheets is pronounced in all shown profiles, the magnetic field magnitude, the component generated by the sheets (it is \( B_y \), the other components are \( B_x = 0 \) and \( B_z = B_0 \)), and the hemispheric particle flux. The latter quantity is plotted for two cases: for the initially isotropic distribution and for the anisotropic distribution with \( A_1 = 1/3 \). These two profiles are quite similar; they have flux depressions on both sides of the sheets. This is due to the fact that the spacecraft meets the same field lines in front of and behind the sheet, so \( B_m \) is the same but \( B_1 \) differs for a particular line (cf. Eq. 28). Therefore the depression repeats, but not exactly with the same profile.

Changing \( y_1 \) to a limiting case \( y_1 = 0.999y_0 \) (approaching case with a uniform current distribution, covered in Romashets and Vandas, 2012), we get what is shown in Fig. 8b. Depressions are deeper because the field magnitude is higher near the edges. The flux variations are very similar in Fig. 8a and b elsewhere.

Figure 8c complements Fig. 8a. The sheets have the same \( y_0/x_0 = 1/2 \) but the non-zero thickness \( d_0/x_0 = 1/4 \) and a
uniform current density, so the solution by Vandas and Romashets (2014) is used. Magnetic field variations are smaller than in Fig. 8a because the current is more diluted. Also the flux variations are smaller and qualitatively different. But if the thickness becomes small, $d_0/x_0 = 1/100$, this case, shown in Fig. 8d, approaches the one displayed in Fig. 8b. Only the flux depressions are not so deep, but they resemble that in Fig. 8a because of the non-zero sheet thickness.

Figure 8e shows the case with the sheets of non-zero thickness and a uniform current, when the sheets are prolate (resemble filaments): $y_0/x_0 = 1/4$ and $d_0/x_0 = 1/2$, i.e., $d_0 > y_0$. In fact, the values of $y_0$ and $d_0$ are swapped from the case of Fig. 8c. The variations become larger in comparison with this figure.

The $B_y$ profiles have a general behavior during the passage of the sheets (Fig. 8a–e): the component has one polarity, it reverses and becomes comparable or larger (in the absolute value) between the sheets, and finally it reverses again. There is a dip between the sheets, which depends (among others) on the sheet separation distance; if it is smaller, the dip is smaller. This behavior is qualitatively in agreement with observations of a disturbed magnetic field component. See for instance Figs. 3 and 4 by Armstrong and Zmuda (1973) or Fig. 2b by Waters et al. (2001). In the last figure, there is no dip in the field. In our model, it occurs when the sheets are in contact, as shown in Fig. 8f (model by Vandas and Romashets, 2014, with $d_0 = x_0$). However, quantitatively this figure is different from Fig. 2b in Waters et al. (2001), for our values of the disturbed component are 1 order of magnitude larger. These high values are necessary for non-negligible variations of particle fluxes along magnetic field lines. They were achieved by a relatively high value of $B_i$ (10% of the background value). That is why we speak of a superstorm (of the Carrington event caliber) when there could be measurable effects on energetic-particle fluxes. In common storm conditions, flux variations would be below 1% and probably masked by other effects.

5 Conclusions

In a previous work (Romashets and Vandas, 2012), Euler potentials of two current sheets of finite width and zero thickness, aligned to a uniform ambient magnetic field, were presented. Electric surface currents were uniformly distributed on the sheets. However, for this system, the magnetic field reaches infinite values at the sheet edges, and it raises the question of its applicability to real systems.

In the present paper, we consider a current distribution in which the current falls to zero at the edges (in a trapezoidal profile). This removes the field divergence while retaining the field configuration very close to the former one. Therefore former results and conclusions of Romashets and Vandas (2012) are valid and applicable to the more realistic model presented in this work. In addition, this is also emphasized by an important conclusion from the new model – that for the current sheet widths larger than their separation ($y_0 < x_0$), magnetic field configuration becomes insensitive to the position where the current starts to drop (value $y_1$ of the model). The model described in this work can serve as a simple model for Birkeland currents, namely as a local model when they have a filamentary structure. We emphasize a local model, because in such an environment simple assumptions as a planar geometry of the sheets and a background homogeneous magnetic field might be justifiable. These simple assumptions are necessary for our analytical approach. And analytical results are much more flexible for calculations when initial parameters change. In principle, the method used here can be applied to a system of local sheets (filaments), but the treatment would be much more complex.

Using Euler potentials we investigated the influence of current sheets of zero and non-zero thicknesses on energetic-particle fluxes and analyzed their characteristic variations near the sheets. The variations might serve as a tool to reveal geometric parameters of the sheets. Figure 8 shows characteristic field and particle-flux profiles for selected geometries. The particle-flux variations in the vicinity of the sheets are pronounced only for very disturbed periods, like in a geomagnetic superstorm.

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