Modeling anisotropic Maxwell–Jüttner distributions: derivation and properties

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Abstract. In this paper we develop a model for the anisotropic Maxwell–Jüttner distribution and examine its properties. First, we provide the characteristic conditions that the modeling of consistent and well-defined anisotropic Maxwell–Jüttner distributions needs to fulfill. Then, we examine several models, showing their possible advantages and/or failures in accordance to these conditions. We derive a consistent model, and examine its properties and its connection with thermodynamics. We show that the temperature equals the average of the directional temperature-like components, as it holds for the classical, anisotropic Maxwell distribution. We also derive the internal energy and Boltzmann–Gibbs entropy, where we show that both are maximized for zero anisotropy, that is, the isotropic Maxwell–Jüttner distribution.

Keywords. Electromagnetics (plasmas) – solar physics astrophysics and astronomy (magnetic fields) – space plasma physics (kinetic and MHD theory)

1 Introduction

The Maxwell–Boltzmann distribution describes classical non-interacting particles at thermal equilibrium. This distribution is generalized within the framework of special relativity, leading to the Maxwell–Jüttner distribution, named after Ferencz Jüttner (1911).

In the case of anisotropic temperature, the Maxwell distribution becomes anisotropic on the kinetic degrees of freedom, which contribute differently in the internal energy of the system (e.g., solar wind: Olsen and Leer, 1999; Feldman et al., 1975; Pilipp et al., 1987; Phillips and Gosling, 1990; Kasper et al., 2002; Matteini et al., 2007; Štverák et al., 2008 – magnetosphere: Pilipp and Morfil, 1976; Renuka and Viswanathan, 1978; Tsurutani et al., 1982; Sckopke et al., 1990; Gary, 1992; Bavassano Cattaneo et al., 2006; Nishino et al., 2007; Cai et al., 2008; Winglee and Harnett, 2016 – also see the corresponding formulations in Krall and Trivelpiece, 1973; Livadiotis and McComas, 2014a; Livadiotis, 2015).

Therefore, we have two main types of generalization of the Maxwell–Boltzmann distribution, that is, the relativistic and the anisotropic description. However, there is no consistent modeling for both the generalizations together, namely, the anisotropic relativistic case, and the formalism of the anisotropic Maxwell–Jüttner distribution remains unknown. It is important to note that such a model may not be unique. From our experience on kappa distributions, a well-defined and unique isotropic model may degenerate to several anisotropic models, each one, however, corresponding to different physical meaning (see Sect. 5 in Livadiotis, 2015).

The purpose of this paper is to develop the anisotropic Maxwell–Jüttner distribution and examine its properties. First, the paper provides the characteristic conditions that a consistent and well-defined anisotropic modeling of the Maxwell–Jüttner distribution needs to fulfill. Then, guided by these conditions, the paper derives such a consistent model and, finally, examines several basic properties related to thermodynamics, e.g., the behavior of the internal energy, temperature, and entropy.

Next, Sect. 2 shows briefly the derivation of the standard, isotropic Maxwell–Jüttner distribution. Then, Sect. 3 proceeds to model the anisotropic Maxwell–Jüttner distribution. First, we provide the characteristic conditions of such a consistent model, and then examine several models, showing their possible advantages and/or failures in accordance with these conditions; finally, we end up with the correct model. In Sect. 4, we examine the properties and thermodynamics of
this model, e.g., the internal energy, the partition of temperature to its directional components, and the entropy. Finally, Sect. 5 summarizes the conclusions.

2 Isotropic Maxwell–Jüttner distribution

The Maxwell–Boltzmann (MB) distribution $P_M$ describes the velocities $\mathbf{u}$ or the kinetic energy $\varepsilon = \frac{1}{2}mu^2$ of the particles at thermal equilibrium, far from the limit of the speed of light, i.e.,

$$P_{MB}(p; \theta) = \left(\frac{\pi m^2 \theta^2}{\Gamma(\frac{1}{2}d)}\right) \cdot \exp\left(-\frac{\pi p^2}{k_B T}\right),$$

(1a)

where $\theta = \sqrt{2k_B T / m}$, $u \ll c$, or, in terms of the kinetic energy,

$$P_{MB}(\varepsilon; T) = \left(\frac{k_B T}{\Gamma(\frac{1}{2}d)}\right) \cdot \exp\left(-\frac{\varepsilon}{k_B T}\right) \cdot \varepsilon^{\frac{1}{2}d-1},$$

(1b)

where $\theta$ is the temperature in speed dimensions, called thermal speed, and $d$ denotes the kinetic degrees of freedom of each particle. (Note that the temperature is defined in the fluid’s rest frame, where the bulk speed $u_b$ is zero. In the non-relativistic case, this can be shown by using $\varepsilon = \frac{1}{2}m(u - u_b)^2$.)

The relativistic generalization of Eq. (1a), that is, the Maxwell–Jüttner (MJ) distribution, is given by

$$P_{MJ}(\gamma) \propto \gamma^2 \beta(\gamma) \cdot e^{-\frac{\varepsilon}{\Theta}}, \quad \Theta \equiv \frac{k_B T}{E_0}, \quad E_0 \equiv mc^2,$$

(2)

where $\beta \equiv \frac{u}{c}$ and $\gamma(\beta) \equiv (1 - \beta^2)^{-\frac{1}{2}}$. (Note that the inverse of the unitless temperature $\Theta$ is the relativistic coldness $\zeta$, Rezzola and Zanotti, 2013.) This distribution (Eq. 2) can be derived as follows.

According to the relativistic formalism for the particle momentum and energy, we have

$$p = mc \cdot \gamma(\beta) \cdot \beta, \quad E(\beta) = \gamma(\beta) \cdot E_0,$$

(3)

while the kinetic energy is given by $\varepsilon = E - E_0 = (\gamma - 1) \cdot E_0$.

The Boltzmann distribution of a Hamiltonian is $P_{MJ}(H) \propto \exp\left[-\frac{H}{k_B T}\right]$. In the absence of a potential energy, $H$ is simply given by the particle energy $E$, thus,

$$P_{MJ}(E) \propto \exp\left(-\frac{E}{k_B T}\right) \propto \exp\left(-\frac{\varepsilon}{\Theta}\right).$$

(4a)

(Note that $E$ is the sum of the kinetic $\varepsilon$ and inertial energy $E_0$.) Then, we include the $d$-dimensional density of states:

$$P_{MJ}(\gamma) \propto \gamma^{d-1} \frac{d p(\gamma)}{d \gamma} \cdot \exp\left(-\frac{\varepsilon}{\Theta}\right)$$

(4b)

so that

$$\int P_{MJ}(p) dp_1 \ldots dp_d \propto \int \exp\left[-\frac{E(p)}{k_B T}\right] dp_1 \ldots dp_d$$

$$= \int \exp\left[-\frac{E(p, \Omega_d)}{k_B T}\right] d\Omega_d p^{d-1} dp$$

$$= \int \exp\left[-\frac{E(\gamma, \Omega_d)}{k_B T}\right] \left[p(\gamma)^{d-1} \frac{d p(\gamma)}{d \gamma}\right] d\Omega_d d\gamma,$$

where $\Omega_d$ denotes the $d$-dimensional solid angle. For isotropic distributions, we have

$$\int P_{MJ}(p) dp_1 \ldots dp_d \propto \int \exp\left[-\frac{E(\gamma)}{k_B T}\right]$$

(5a)

$$\times \left[p(\gamma)^{d-1} \frac{d p(\gamma)}{d \gamma}\right] d\Omega_d d\gamma \equiv \int d\Omega_d \cdot \int P_{MJ}(\gamma) d\gamma,$$

or

$$P_{MJ}(\gamma) \propto \exp\left[-\frac{E(\gamma)}{k_B T}\right] \cdot p(\gamma)^{d-1} \frac{d p(\gamma)}{d \gamma}.$$ 

(5b)

Then, $d(\gamma \beta) = \gamma(\gamma - 1)^{-\frac{1}{2}} d\gamma = \beta^{-1} d\gamma$ so that

$$p(\gamma)^{d-1} \frac{d p(\gamma)}{d \gamma} = (mc)^d(\gamma \beta)^{d-1} \frac{d (\gamma \beta)}{d \gamma}$$

(6)

$$= (mc)^d \gamma^{-d} \beta^{-d-2},$$

or

$$P_{MJ}(\gamma) \propto \gamma^{d-1} \beta^{-d-2} \cdot e^{-\frac{\varepsilon}{\Theta}} \propto \gamma(\gamma^2 - 1)^{\frac{d}{2} - 1} \cdot e^{-\frac{\varepsilon}{\Theta}}$$

(7)

because $\frac{E}{k_B T} = \frac{\varepsilon}{\Theta}$. Then, we normalize the distribution Eq. (7). We set

$$P_{MJ}(p; \Theta) dp_1 dp_2 \ldots dp_d$$

(8)

$$= N \cdot e^{-\frac{\varepsilon}{\Theta}} dp_1 dp_2 \ldots dp_d,$$

and the angular integration,

$$dp_1 dp_2 \ldots dp_d = B_d p^{d-1} dp$$

$$= \frac{1}{2} B_d (mc)^d \left[\left(\frac{p}{mc}\right)^2\right]^{\frac{d}{2} - 1} d\left(\frac{p}{mc}\right)^2,$$

where $B_d = \frac{2\pi \frac{d}{2}}{\Gamma\left(\frac{d}{2}\right)}$ is the surface of the unit $d$-dimensional sphere. Then, using the identity $\gamma^2 = (\frac{p}{mc})^2 + 1$, we have

$$P_{MJ}(p; \Theta) dp_1 dp_2 \ldots dp_d$$

(9)

$$= N \cdot \frac{1}{2} B_d (mc)^d \cdot e^{-\frac{\varepsilon}{\Theta}} (\gamma^2 - 1)^{\frac{d}{2} - 1} d(\gamma^2 - 1),$$

and
Therefore, the normalized distribution is

\[ Z = \pi^{d/2} (mc)^d \cdot \Theta^{d/2} K_{d+1}^{(1)} \left( \frac{1}{\Theta} \right) . \quad (14c) \]

The inverse of the normalization constant gives the partition function \( Z \equiv \frac{1}{N} : \)

\[ Z = \pi^{d/2} (mc)^d \cdot \Theta^{d/2} K_{d+1}^{(1)} \left( \frac{1}{\Theta} \right) . \quad (14c) \]

Therefore, the normalized distribution is

\[ P_{\text{MJ}}(p; \Theta) \, dp_1 \, dp_2 \ldots dp_d \]

\[ = \pi^{d/2} (mc)^d \cdot \Theta^{d/2} K_{d+1}^{(1)} \left( \frac{1}{\Theta} \right)^{-1} . \quad (15a) \]

\[ \times \exp \left[ -\gamma(p)/\Theta \right] \, dp_1 \, dp_2 \ldots dp_d , \]

or we may derive the normalized distribution in terms of \( \gamma , \)

\[ P_{\text{MB}}(\gamma ; \Theta) d\gamma \]

\[ = \pi^{d/2} \frac{1}{2} \frac{1}{\Theta^d} \cdot \Theta^{d/2} K_{d+1}^{(1)} \left( \frac{1}{\Theta} \right)^{-1} \cdot e^{-\gamma (p^2 - 1)^{\frac{2}{d}} - 1} \gamma d\gamma . \quad (15b) \]

(Note that in Sect. 4.4 we show that the parameter \( \Theta \) coincides with the thermodynamic definition of temperature.)

In Fig. 1 we plot the MJ distribution of the kinetic energy \( E_k = E_0(\gamma - 1) \) for various temperatures and dimensionalities, which is given by \( P(E_k) = P_{\text{MB}} \left( \gamma = 1 + \frac{E_k}{E_0} \right) \). We observe that the MJ distribution is well-approximated by the classical MB distribution, while the differences between the two distributions are more clear for \( T \sim 1 \text{ MeV} \).

When the MB distribution clearly deviates from the MJ distribution of the same temperature and dimensionality, then a different MB distribution can give a good approximation to the MJ distribution. This new MB distribution can be either (i) a convected MB distribution, that is, an MB distribution with the same dimensionality, but with different temperature \( T_{\text{MB}} \) and bulk speed \( u_b \) (or bulk energy \( E_b \equiv \frac{1}{2} m u_b^2 \)), or (ii) an MB distribution with the same bulk speed, but with different temperature \( T_{\text{MB}} \) and degrees of freedom \( d_{\text{MB}} \). These two types of approximations of the MJ distribution by an MB distribution are illustrated in Fig. 2.
Also useful is the expression of the distribution in the velocity space (Dunkel et al., 2007). Given that \( \frac{d(p)}{dp} \propto \gamma^3 \), we find

\[
dp_1 \ldots dp_d = p^{d-1} dp d\Omega_d
\]

\[
= (mc)^d \gamma^{d-1} b^{d-1} \frac{d(\gamma)}{d\beta} d\beta d\Omega_d
\]

\[
= (mc)^d \gamma^{d+2} b^d d\beta d\Omega_d
\]

\[
= (mc)^d \gamma^{d+2} b_1 \ldots d\beta_d.
\]

hence

\[
P_M(\beta; \Theta) d\beta_1 d\beta_2 \ldots d\beta_d \propto \pi^{-\frac{d}{2}} 2^{-\frac{d+1}{2}} \Theta^{\frac{d}{2}} K_{\frac{d+1}{2}} \left( \frac{1}{\Theta} \right)^{-1}
\]

\[
\times \exp \left[ -\frac{\gamma(\beta)}{\Theta} \right] \frac{1}{\Theta} \gamma^{d+2} b_1 \ldots d\beta_d.
\]

For example, for \( d = 3 \) we have

\[
P_M(\gamma; \Theta) d\gamma = \frac{1}{4\pi} (mc)^{-3} \Theta K_2 \left( \frac{1}{\Theta} \right)^{-1} \cdot e^{-\frac{\gamma}{\Theta}} d\gamma
\]

and

\[
P_M(\beta; \Theta) d\beta = \frac{4}{\pi} \Theta K_2 \left( \frac{1}{\Theta} \right)^{-1} \cdot e^{-\frac{\gamma}{\Theta}} \gamma^{d+2} d\beta_1 \ldots d\beta_d.
\]

Figure 2. The plot of an MJ distribution of temperature \( T \) and degrees of freedom \( d \) can be misinterpreted either by (i) a convected MB distribution, that is, with the same dimensionality, but with different temperature \( T_{MB} \) and bulk speed \( u_b \) (or energy \( E_b \equiv \frac{1}{2} m u_b^2 \)) or (ii) an MB distribution with the same bulk speed, but with different temperature \( T_{MB} \) and degrees of freedom \( d_{MB} \). (a) A co-plot of an MJ distribution and the two mentioned MB distributions. (b) The temperature \( T_{MB} \) and the bulk energy \( E_b \) are plotted as functions of \( T \). (c) The temperature \( T_{MB} \) as a function of \( T \), for various dimensionalities, and (d) the dimensionality \( d_{MB} \) as a function of the dimensionality \( d \) and various temperatures.

2. Non-symplectic energy–temperature components:

   a. The inertial mass energy does not depend on kinetic terms and is isotropic.

   b. Each kinetic component is connected with the corresponding temperature component.

3. Temperature partition to its anisotropy components:

   a. Internal energy must be additive to its components,

   \[
   U = \frac{3}{2} k_B T, \quad T = \frac{1}{3} \sum_{i=1}^{d} T_i.
   \]

   b. Internal energy and entropy is maximized for zero anisotropy.

Next, we are going to derive and criticize several models using the above conditions.

3.2 Model: square inverse temperature

The particle energy–momentum relation can be a motive for a certain model of the anisotropic MJ distribution. Indeed, following the derivation of the energy–momentum relation,

\[
\left( \frac{1}{c} E, \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \right) \equiv \diag(+1, -1, -1) \cdot \left( \frac{1}{c} E, \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \right)
\]

\[
= (mc, 0)^t \cdot (mc, 0) \Rightarrow E^2 - c^2 p^2 = (mc^2)^2,
\]

we may write

3 Anisotropic Maxwell–Jüttner distribution

3.1 General characteristic conditions for consistent modeling

The derived distribution must have the following characteristic conditions.

1. Correspondence at both classical and isotropic limits:

   a. Correspondence for \( c \to \infty \) (anisotropic MB).

   b. Correspondence for \( T \to 1 \) (isotropic MJ).


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where the inertial energy is kept isotropic; i.e., the anisotropy characterizes only the momenta (according to the conditions in Sect. 3.1). Then, we may write Eq. (18) as

\[
E^2 - \sum_{i=1}^{3} \left( \frac{T}{T_i} \right)^2 c^2 p_i^2 = (mc^2)^2, \quad \text{or},
\]

\[
\frac{E}{k_B T} = \Theta^{-1} \left[ 1 + \frac{3}{T} \sum_{i=1}^{N} \eta_i \left( \frac{p_i}{mc} \right)^2 \right],
\]

where \( \eta_i \equiv \frac{T}{T_i} \) stands for the directional anisotropy.

The problem with the above derivation is that we are violating the energy–momentum relation of the free particle. \( E(p) = \sqrt{c^4 p^2 + (mc^2)^2} \), or \( E(p) \approx \frac{p^2}{2m} \) (for \( \beta \ll 1 \)), which constitute isotropic relations. Features from statistical mechanics should not be involved in one-particle (non-statistical) relations; in contrast, one-particle relations can certainly be examined by statistical mechanics. Nevertheless, we may obtain the same relation under different derivation path and physical interpretation.

The kinetic degrees of freedom, \((d \cdot N) = \sum_{m=1}^{M} (d \cdot N)_m \), are separated into \( M \) groups (particle subsystems of different temperature-like components): \((d \cdot N)_1 \) of them have temperature component \( T_1 \) (group \( m = 1 \)), \((d \cdot N)_2 \) of them have temperature component \( T_2 \) (group \( m = 2 \)), \ldots , \((d \cdot N)_M \) of them have temperature component \( T_M \) (group \( m = M \)); then, the probability of the Hamiltonian factor becomes

\[
\exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_i}{k_B T_i} \right) \rightarrow \exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_i}{k_B T_i} \right)
\]

\[
\times \exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_{i2}}{k_B T_2} \right), \ldots , \exp \left( -\sum_{i=1}^{N} \frac{E_{iM}}{k_B T_M} \right),
\]

where \( E_i \) constitutes the energy component that depends solely on the \( i \)th kinetic degree of freedom. The temperature-like components do not constitute the unique temperature of the system that is given by \( T \); they may be called temperatures though, for simplicity. Typically, the particle dimensionality determines the number of the groups, \( M \leq d \). Some examples are, when each dimension has a different temperature, \( M = d \),

\[
\exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_i}{k_B T_i} \right) \rightarrow \exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_i}{k_B T_i} \right)
\]

\[
\times \exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_{i2}}{k_B T_2} \right), \ldots , \exp \left( -\sum_{i=1}^{N} \frac{E_{iM}}{k_B T_M} \right),
\]

or when we have anisotropy at one direction, e.g., \( d = 3 \), where one degree of freedom is parallel to a certain direction (e.g., that of the magnetic field), and two degrees of freedom are perpendicular to that, i.e.,

\[
\exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_i}{k_B T_i} \right) \rightarrow \exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_i}{k_B T_i} \right)
\]

\[
\times \exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_{i2}}{k_B T_2} \right), \ldots , \exp \left( -\sum_{i=1}^{N} \frac{E_{iM}}{k_B T_M} \right).
\]

Therefore, each group is characterized by the same one-particle relation but different statistics. For instance, in the previous examples, the one-particle distribution will be, respectively,

\[
P_{MB}(E_1, \ldots, E_d; T_1, \ldots, T_d) \propto \exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_i}{k_B T_i} \right) \text{ and } \]

\[
P_{MB}(E_\perp, E_\parallel; T_\perp, T_\parallel) \propto \exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_i}{k_B T_i} \right).
\]

The anisotropy applies by considering different statistics for each component (or group); hence, while the temperature \( T \) characterizes the whole particle system and its internal energy, different temperature-like components characterize each of the kinetic degrees of freedom (non-symplectic property in Sect. 3.1),

\[
E = \sum_{i=1}^{d} E_i \Leftrightarrow \frac{E}{k_B T} \rightarrow \sum_{i=1}^{d} \frac{E_i}{k_B T_i}.
\]

Namely, in the isotropic MB distribution the independence of the probabilities \( P(E_i; T_i) \) leads to the summation of the partial energies \( E_i \), but in the anisotropic case it leads to the summation of \( E_i/k_B T_i \), i.e.,

\[
P_{MB}(E_1, \ldots, E_d; T_1, \ldots, T_d) \propto \prod_{i=1}^{d} P(E_i; T_i) \]

\[
= \prod_{i=1}^{d} \exp \left( \frac{E_i}{k_B T_i} \right) = \exp \left( -\sum_{i=1}^{\frac{d}{N}} \frac{E_i}{k_B T_i} \right).
\]

In the relativistic case, the probabilities are not independent, so their product cannot lead to the joint distribution \( P_{MB} \).

\[
P_{MB}(E_1, \ldots, E_d; T_1, \ldots, T_d) \neq N \prod_{i=1}^{d} P(E_i; T_i) \]
(N: normalization constant). The model of square inverse temperature in Eq. (18), involves a square summation, in contrast to the linear model in Eq. (23), i.e.,

\[
E = \sum_{i=1}^{d} E_i^2 \rightarrow E \rightarrow \sqrt{\frac{d}{k_B T_i}} \cdot (p_i c)^2 \cdot \frac{T_i}{T}^2.
\]

Let \( E_0 = m c^2 \) be the inertial energy and \( E_i = c p_i \), the \( i \)-th momentum component expressed in energy units. Then, we may use the \( L_q \)-type summation (e.g., Livadiotis, 2007),

\[
A_1 \oplus_q A_2 \oplus_q \cdots \oplus_q A_d = \left( A_1^q + A_2^q + \cdots + A_d^q \right)^{\frac{1}{q}}
\]

(27)

According to this, the energies \( E_0, E_1, \ldots, E_d \) are combined under an \( L_q \)-type summation to give

\[
E \rightarrow \left[ \frac{E_0^q}{(k_B T_0)^q} + \sum_{i=1}^{d} \frac{E_i^q}{(k_B T_i)^q} \right]^{\frac{1}{q}}
\]

(28a)

that is,

\[
E \rightarrow \frac{E_0}{k_B T_0} + \sum_{i=1}^{d} \frac{E_i}{k_B T_i}, \quad \text{for } q = 1, \text{ or}
\]

\[
E \rightarrow \frac{E_0^2}{(k_B T_0)^2} + \sum_{i=1}^{d} \frac{E_i^2}{(k_B T_i)^2}, \quad \text{for } q = 2.
\]

(28b)

Again, we reiterate that the inertial energy \( E_0 \) is common in all directions and it does not contribute to the thermal anisotropy so that its temperature directional component equals the total temperature, \( T_0 = T \),

\[
P(E_0, E_1, \ldots, E_d; T, T_1, \ldots, T_d)
\]

\[
\propto \exp \left[ - \frac{E_0^2}{(k_B T_0)^2} + \sum_{i=1}^{d} \frac{E_i^2}{(k_B T_i)^2} \right]^{\frac{1}{q}}
\]

(29)

(\( \text{Note that the temperature } T \text{ is a function of the directional temperature-like components } T_i, \text{ thus, it may be excluded from the independent components of the probability distribution } P; \text{ the same can be done for the inertial energy } E_0, \text{ as it is not a variable for particles of a certain mass.)} \)

\[
P(p; T) \propto \exp \left[ - \frac{E_0}{k_B T} \cdot \left( 1 + \sum_{i=1}^{d} \left( \frac{T_i}{T} \right)^2 \cdot (p_i c)^2 \right) \right].
\]

(30)

where we use the vector-like notation \( T = (T_1, \ldots, T_d) \). The disproof of this formalism comes from the failure to recover the anisotropic MB distribution. Indeed,

\[
P(p; T) \propto \exp \left[ - \frac{E_0}{k_B T} \cdot \left( 1 + \sum_{i=1}^{d} \left( \frac{T_i}{T} \right)^2 \cdot (p_i c)^2 \right) \right]
\]

\[
\equiv \exp \left[ - \sum_{i=1}^{d} \left( \frac{T_i}{T} \cdot \frac{1}{(k_B T_i)^2} \right) \right].
\]

(31)

which differs from the expected

\[
P(p; T) \propto \exp \left[ - \sum_{i=1}^{d} \left( \frac{1}{(k_B T_i)^2} \right) \right].
\]

(32)

\[
\text{3.3 Model: symplectic inverse temperature}
\]

It is always possible to write the anisotropic relation in a symplectic way so that both correspondence conditions hold (Sect. 3.1). In the following example, the inertial energy is characterized by anisotropy, while there is one symplectic kinetic component \((i = 2)\):

\[
E \rightarrow \frac{E_0}{k_B T} \cdot \left( 1 + \sum_{i=1}^{d} \left( \frac{T_i}{T} \right)^2 \cdot (p_i c)^2 \right)
\]

\[
\propto \exp \left[ - \sqrt{ \frac{E_0^2}{(k_B T_0)^2} + \frac{E_1^2}{(k_B T_1)^2} + \frac{E_2^2}{(k_B T_2)^2} } \right]
\]

(33a)

and thus the distribution becomes

\[
P(p; T) \propto \exp \left[ - \sqrt{ \frac{E_0^2}{(k_B T_0)^2} + \frac{(p_1 c)^2}{(k_B T_1)^2} + \frac{(p_2 c)^2}{(k_B T_2)^2} } \right].
\]

(33b)

Therefore, the correspondence conditions are fulfilled; i.e., the Hamiltonian factor is approximated for \( c \rightarrow \infty \) to

\[
P(p; T) \propto \exp \left[ - \sum_{i=1}^{2} \left( \frac{1}{(k_B T_i)^2} \right) \right],
\]

(33c)

which is the anisotropic MB distribution. However, the symplectic property is not fulfilled (Sect. 3.1).

In another example, for any value of \( b \), we model the Hamiltonian factor as

\[
E \rightarrow \frac{E_0^2}{(k_B T_0)^2 (T_1 T_2 / T^2)^{2b}} + \frac{E_0^2}{(k_B T_1) (k_B T_2) (T_1 T_2 / T^2)^b},
\]

(34a)
and thus the distribution becomes
\[ P(p; T) \propto \exp \left\{ -\frac{E_0^2}{(k_B T)^2 (T_1 T_2/T^2)^2} + \frac{(p_0 c)^2}{(k_B T_1) (k_B T_2) (T_1 T_2/T^2)^2} \right\} \] (34b)

The Hamiltonian factor is approximated for \( c \rightarrow \infty \) to
\[ \left[ \frac{E_0^2}{(k_B T)^2 (T_1 T_2/T^2)^2} + \frac{(p_0 c)^2}{(k_B T_1) (k_B T_2) (T_1 T_2/T^2)^2} \right] \]
\[ + \frac{(p_2 c)^2}{(k_B T_1) (k_B T_2) (T_1 T_2/T^2)^2} \] (34c)

so that
\[ P(p; T) \propto \exp \left\{ -\sum_{i=1}^{2} \left( \frac{1}{2k_B T_i} \right) \right\} . \] (34d)

For example, for \( b = \frac{1}{2} \), we obtain
\[ P(p; T) \propto \exp \left\{ -\frac{E_0^2}{(k_B T)^2 (T_1 T_2/T^2)^2} + \frac{(p_0 c)^2}{(k_B T_1) (k_B T_2) (T_1 T_2/T^2)^2} \right\} \]
\[ + \frac{(p_2 c)^2}{(k_B T_1) (k_B T_2) (T_1 T_2/T^2)^2} . \] (35)

For \( b = 0 \), we derive the only anisotropic distribution that obeys the correspondence and also the symplectic conditions in Sect. 3.1, which is examined as follows:
\[ P(p; T) \propto \exp \left\{ -\frac{E_0^2}{(k_B T)^2} + \frac{(p_0 c)^2}{(k_B T_1) (k_B T_2) (T_1 T_2/T^2)^2} \right\} \] (36)

Let us now suggest a model that fulfills all the requirements given in Sect. 3.1.

3.4 Suggested model: linear inverse temperature

We may write the energy–momentum relation in a way that the summation on the energy component is linear, namely,
\[ E = \sqrt{E_0^2 + \sum_{i=1}^{d} c^2 p_i^2} = \sqrt{E_0 \cdot \left( E_0 + \sum_{i=1}^{d} \frac{p_i^2}{m} \right)} \]
\[ = \sqrt{E_0 \cdot \left( E_0 + \sum_{i=1}^{d} E_i \right)} . \] (37a)

Hence, the Hamiltonian factor becomes
\[ \frac{E}{k_B T} \rightarrow \sqrt{\frac{E_0}{k_B T} \left( \frac{E_0}{k_B T} + \sum_{i=1}^{d} \frac{E_i}{k_B T_i} \right)} \] (37b)

where \( E_0 = mc^2 \) and \( E_i = \frac{1}{m} p_i^2 \). Then, the distribution becomes
\[ P(p; T) \propto \exp \left\{ -\frac{E_0}{k_B T} \sqrt{1 + \sum_{i=1}^{d} \frac{T_i}{T_0} \left( \frac{p_i}{mc} \right)^2} \right\} \] (37c)

– Correspondence for \( c \rightarrow \infty \) (anisotropic MB):
\[ P(p; T) \propto \exp \left\{ -\frac{E_0}{k_B T} \sqrt{1 + \sum_{i=1}^{d} \frac{T_i}{T_0} \left( \frac{p_i}{mc} \right)^2} \right\} \]
\[ - \exp \left( -\frac{E_0}{k_B T} - \sum_{i=1}^{d} \frac{1}{2k_B T_i} \right) \propto \exp \left( -\frac{\gamma}{\Theta} \right) . \]

– Correspondence for \( T_i/T \rightarrow 1 \) (isotropic MJ):
\[ P(p; T) \propto \exp \left\{ -\frac{E_0}{k_B T} \sqrt{1 + \left( \frac{p}{mc} \right)^2} \right\} \propto \exp \left( -\frac{\gamma}{\Theta} \right) . \]

The normalized anisotropic Maxwell–Jüttner distribution is
\[ P(p; T) = N \cdot \exp \left\{ -\frac{E_0}{k_B T} \sqrt{1 + \sum_{i=1}^{d} \frac{T_i}{T_0} \left( \frac{p_i}{mc} \right)^2} \right\} \] (38a)
or
\[ P(p; \Theta) = N \cdot \exp \left\{ -\frac{\Theta}{\Theta} \sqrt{1 + \sum_{i=1}^{d} \frac{\Theta_i}{\Theta} \left( \frac{p_i}{mc} \right)^2} \right\} , \]
with normalization
\[ N^{-1} = \int_{-\infty}^{\infty} \exp \left\{ -\frac{E_0}{k_B T} \sqrt{1 + \sum_{i=1}^{d} \frac{T_i}{T_0} \left( \frac{p_i}{mc} \right)^2} \right\} dp_1 \ldots dp_d \]
\[ = B_d \left( \frac{E_0}{k_B T} \right)^{\frac{1}{2d}} \prod_{i=1}^{d} \left( \frac{k_B T_i}{E_0} \right)^{\frac{1}{2d}} \left( \frac{mc}{p_i} \right)^{d} \]
\[ \times \int_{0}^{\infty} e^{-\frac{E_0}{k_B T} \sqrt{1 + \frac{T_i}{T_0} \left( \frac{p_i}{mc} \right)^2}} \sqrt{1 + \frac{p_i}{mc}} \frac{dp_i}{p_i} \]
\[ = \frac{1}{2} B_d \left( \frac{E_0}{k_B T} \right)^{\frac{1}{2d}} \prod_{i=1}^{d} \left( \frac{k_B T_i}{E_0} \right)^{\frac{1}{2d}} \left( \frac{mc}{p_i} \right)^{d} \]
\[ \times \int_{1}^{\infty} e^{-\frac{E_0}{\gamma^2 \beta^2}} \, (\gamma^2 - 1)^{\frac{d-1}{2}} \, d(\gamma^2 - 1) \]

\[ = \frac{1}{2} B_d \left( \frac{E_0}{k_B T} \right)^{\frac{1}{2} + \frac{d}{2}} \prod_{i=1}^{d} \left( \sqrt{\frac{k_B T_i}{E_0}} \right) (mc)^d \cdot I_d. \]

where we set \( \gamma = \sqrt{1 + \frac{p^2}{\beta^2}} \) and \( \tilde{p}^2 = \sum_{i=1}^{d} \left( \frac{T_i}{T} \right)^2 \left( \frac{1}{mc} \right)^2 \).

Hence, we have

\[ N = \pi^{\frac{1-d}{2}} 2^{-\frac{d+1}{2}} (mc)^{-d} \cdot \left( \frac{k_B T}{E_0} \right)^{\frac{1}{2}} \prod_{i=1}^{d} \left( \frac{k_B T_i}{E_0} \right)^{-\frac{1}{2}} \left( \frac{E_0}{k_B T} \right)^{-1}. \]

(38b)

Note that for the specific case where \( d = 3 \) and \( \{ T_i \}_{i=1}^{d} = (T_{ii}, T_{jj}, T_{kk}) \) the normalization constant in Eq. (38b) coincides with that derived by Treumann and Baumjohann (2016). Then, the normalized distribution is

\[ P(p; T)dp_{1}...dp_{d} = \pi^{\frac{1-d}{2}} 2^{-\frac{d+1}{2}} \left( \frac{k_B T}{E_0} \right)^{\frac{1}{2}} \prod_{i=1}^{d} \left( \frac{k_B T_i}{E_0} \right)^{-\frac{1}{2}} \left( \frac{E_0}{k_B T} \right)^{-1} \times \exp \left[ -\frac{E_0}{k_B T} \cdot \sqrt{1 + \sum_{i=1}^{d} \frac{T_i}{T} \left( \frac{p_i}{mc} \right)^2} \right] dp_{1}...dp_{d}, \]

or, in terms of \( \Theta_i \),

\[ P(p; \Theta)dp_{1}...dp_{d} = \pi^{\frac{1-d}{2}} 2^{-\frac{d+1}{2}} \Theta_1^2 \prod_{i=1}^{d} \left( \frac{k_B T_i}{E_0} \right)^{-\frac{1}{2}} \left( \frac{E_0}{k_B T} \right)^{-1} \times \exp \left[ -\frac{E_0}{k_B T} \cdot \sqrt{1 + \sum_{i=1}^{d} \Theta_i \left( \frac{p_i}{mc} \right)^2} \right] dp_{1}...dp_{d}. \]

It is useful to write the distribution also in terms of the following arguments:

\[ \tilde{p}_i = \frac{\Theta_i p_i}{\Theta_1 mc}, \quad \tilde{p}^2 = \sum_{i=1}^{d} \tilde{p}^2_i, \quad \gamma \equiv \sqrt{1 + \tilde{p}^2}, \quad \text{thus}, \quad \gamma^2 = \gamma^2 \]

\[ \Theta^{-\frac{1}{2}} \left( \frac{k_B T_i}{E_0} \right)^{-\frac{1}{2}} \left( \frac{E_0}{k_B T} \right)^{-1} \times \exp \left[ -\frac{E_0}{k_B T} \cdot \sqrt{1 + \sum_{i=1}^{d} \Theta_i \left( \frac{p_i}{mc} \right)^2} \right] dp_{1}...dp_{d}, \]

(40)

and

\[ P(\tilde{p}; \Theta)dp_{1}...dp_{d} = \pi^{\frac{1-d}{2}} 2^{-\frac{d+1}{2}} \Theta^{-\frac{1}{2}} \left( \frac{k_B T_i}{E_0} \right)^{-\frac{1}{2}} \left( \frac{E_0}{k_B T} \right)^{-1} \times \exp \left[ -\frac{E_0}{k_B T} \cdot \sqrt{1 + \sum_{i=1}^{d} \frac{T_i}{T} \left( \frac{p_i}{mc} \right)^2} \right] dp_{1}...dp_{d}. \]

(41a)

which coincides with the isotropic MJ distribution in Eq. (15b).

- Correspondence for \( T_i / T \to 1 \) (isotropic MJ):

\[ P(p; \Theta)dp = \int_{\Theta}^{\infty} P(p; \Theta)dp_{1}...dp_{d} \]

\[ = B_d \left( \frac{E_0}{k_B T} \right)^{\frac{1}{2} + \frac{d}{2}} \prod_{i=1}^{d} \left( \frac{k_B T_i}{E_0} \right)^{-\frac{1}{2}} \left( \frac{E_0}{k_B T} \right)^{-1} \times \exp \left[ -\frac{E_0}{k_B T} \cdot \sqrt{1 + \sum_{i=1}^{d} \frac{T_i}{T} \left( \frac{p_i}{mc} \right)^2} \right] dp_{1}...dp_{d}. \]

\[ = \pi^{\frac{1-d}{2}} 2^{-\frac{d+1}{2}} \Theta^{-\frac{1}{2}} \left( \frac{k_B T_i}{E_0} \right)^{-\frac{1}{2}} \left( \frac{E_0}{k_B T} \right)^{-1} \times \exp \left[ -\frac{E_0}{k_B T} \cdot \sqrt{1 + \sum_{i=1}^{d} \frac{T_i}{T} \left( \frac{p_i}{mc} \right)^2} \right] dp_{1}...dp_{d}. \]

\[ \times \exp \left[ -\frac{E_0}{k_B T} \cdot \sqrt{1 + \sum_{i=1}^{d} \frac{T_i}{T} \left( \frac{p_i}{mc} \right)^2} \right] dp_{1}...dp_{d}. \]

(41b)

which is the anisotropic MB distribution, where \( \left( \sqrt{\frac{\pi}{2} \frac{k_B T_i}{E_0}} \left( \frac{k_B T_i}{E_0} \right)^{-1} \exp \left[ -\frac{E_0}{k_B T_i} \right] \right) \approx 1 + O \left( \frac{k_B T_i}{E_0} \right) \). Figure 3 shows the cases of anisotropic/isotropic MJ and MB distributions.
4 Properties and thermodynamics

4.1 Internal energy

We start with the isotropic MJ distribution. The internal energy, $U$, is derived by the mean energy:

\[
\frac{U}{E_0} = \langle \gamma \rangle = \int P_{MI}(\gamma; \Theta) \gamma \, d\gamma
\]

\[
= \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{d}{2}\right)} \cdot \Theta^{\frac{1}{2} - d} \cdot K_{d+1} \left(\frac{1}{\Theta}\right)^{-1} \left(\frac{d}{2} - 1\right)
\]

\[
\times \int_1^\infty e^{-\gamma} (\gamma^2 - 1)^{\frac{d}{2} - 1} \gamma^2 \, d\gamma
\]

\[
= \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{d}{2}\right)} \cdot \Theta^{\frac{1}{2} - d} \cdot K_{d+1} \left(\frac{1}{\Theta}\right)^{-1} \left(\frac{d}{2} - 1\right)
\]

\[
\times \left[d + \frac{1}{2} K_{d+1} \left(\frac{1}{\Theta}\right)ight],
\]

where, according to Eq. (11), we have

\[
I_d + I_{d-2} = \Gamma\left(\frac{d}{2}\right) \frac{1}{2} \cdot \Theta^{\frac{1}{2} - d} \cdot K_{d+1} \left(\frac{1}{\Theta}\right)^{-1}
\]

\[
\times \left[1 + \frac{1}{2} K_{d+1} \left(\frac{1}{\Theta}\right)\right].
\]

Hence,

\[
\frac{U}{E_0} = d \cdot \Theta + \frac{K_{d+1} \left(\frac{1}{\Theta}\right)}{K_{d+1} \left(\frac{1}{\Theta}\right)},
\]

or

\[
\frac{U}{E_0} = E_0 + d \cdot k_B T + E_0 \left[\frac{K_{d+1} \left(\frac{1}{\Theta}\right)}{K_{d+1} \left(\frac{1}{\Theta}\right)} - 1\right]
\]

\[
\approx E_0 + d \frac{k_B T}{2} + k_B T \cdot O\left(\frac{k_B T}{E_0}\right),
\]

where

\[
\left[\frac{K_{d+1} \left(\frac{1}{\Theta}\right)}{K_{d+1} \left(\frac{1}{\Theta}\right)}\right] \approx 1 - d \cdot \Theta + O(\Theta^2).
\]

In the anisotropic case, the corresponding integral cannot be solved analytically,

\[
\frac{U}{E_0} = \langle \gamma \rangle = \int P_{MI}\left(\vec{\gamma}; \Theta\right) \gamma \, d\vec{\gamma}
\]

\[
= \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{d}{2}\right)} \cdot \Theta^{\frac{1}{2} - d} \cdot K_{d+1} \left(\frac{1}{\Theta}\right)^{-1}
\]

\[
\times \int_1^\infty e^{-\vec{\gamma}} (\vec{\gamma}^2 - 1)^{\frac{d}{2} - 1} \vec{\gamma} \, d\vec{\gamma}
\]
becomes independent of the anisotropy only at the classical limit of $\Theta \to 0$, i.e., for $c \to \infty$, the internal energy becomes independent of the temperature directional components $\Theta_i$.

where $l_i = \left(\frac{\hat{p}_i}{p_i}\right)^2$ is the square of the direction cosine at the $i$th axis. For example, for $d = 3$, we have $l_x = \sin^2 \vartheta \cos^2 \varphi$, $l_y = \sin^2 \vartheta \sin^2 \varphi$, $l_z = \cos^2 \vartheta$. Hence,

$$\frac{U}{E_0} = \Theta^{-\frac{3}{2}} K_2 \left(\frac{1}{\Theta}\right)^{-1} \int_1^\infty \ldots d\gamma d\cos \vartheta d\varphi \tag{45a}$$

$$\ldots \equiv \sqrt{(\Theta + (\gamma^2 - 1) (\Theta \cos^2 \vartheta + \sin^2 \vartheta)) (\Theta \cos^2 \varphi + \Theta \sin^2 \varphi) \cdot e^{-\frac{\gamma \sqrt{2}}{2} (\gamma^2 - 1)^{\frac{1}{4}}}}. \tag{46a}$$

In the case of the parallel/perpendicular anisotropy, we obtain

$$\frac{1}{E_0} U (\Theta_\perp, \Theta_\parallel) = \frac{3 K_2 \left(\frac{3}{\sqrt{6}} \frac{\Theta_\parallel}{\Theta_\perp}\right)^{-1}}{\left(\Theta_\parallel + 2 \Theta_\perp\right) \frac{3}{2}} \left(1 \right) \frac{1}{d} \sum_{i=1}^{d} \Theta_i \left(l_i \right) = \frac{\Theta_\perp}{\Theta_\parallel} \left(l_\parallel \right) \tag{47}$$

This shows the additivity of the temperature directional components characterizing each dimension, as required by the conditions in Sect. 3.1.

4.3 Entropy

The Boltzmann–Gibbs entropic formulation, which is aligned with the classical kinetic theory and the Maxwell distribution (Livadiotis and McComas, 2009), is better known for the discrete probability distribution $\{p_k\}_{k=1}^W$, that is,

$$S(p_k) = - \sum_{k=1}^W p_k \ln p_k. \tag{48a}$$

In the continuous description, however, this is given by (Livadiotis, 2014)

$$\int_{-\infty}^\infty \left[ P_{M/J} (\mathbf{p}; \Theta) \right] \frac{dp_1 \ldots dp_d}{\sigma^d} = 1, \tag{48b}$$

where the momentum scale $\sigma$ is expressed in terms of the Planck constant $h$ (and the plasma density $n$ and temperature $T$ or $\theta \equiv \sqrt{\frac{2mT}{n}}$), derived within a semi-classical statistical/quantum mechanical approach (Livadiotis and McComas, 2013; Livadiotis, 2014):

$$\sigma \equiv \left(\frac{h}{e}\right)^{\frac{1}{2}}. \tag{49}$$
(Note that collisionless plasmas may be characterized by a different quantization constant, Livadiotis and McComas, 2013, 2014b; Livadiotis, 2016; Livadiotis and Desai, 2016.)

Having introduced the momentum scale \( \sigma \), this is included in the expression of the normalization constant \( N \), or the partition function \( Z \equiv 1/N \); then, we have the distribution function as

\[
P_{MJ}(p; \Theta) = N(\Theta) \cdot \exp \left[ -\frac{1}{\Theta} \cdot \sqrt{1 + \sum_{i=1}^{d} \Theta \left( \frac{p_i}{mc} \right)^2} \right],
\]

\[
N(\Theta) \equiv \pi^{\frac{d-1}{2}} \cdot 2^{\frac{d+1}{2}} \left( \frac{\sigma}{mc} \right)^d \times \Theta^{\frac{d}{2}} \prod_{i=1}^{d} \Theta_i^{-\frac{1}{2}} K_{\frac{d+1}{2}} \left( \frac{1}{\Theta} \right)^{-1},
\]

the partition function as

\[
Z = \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{\Theta} \cdot \sqrt{1 + \sum_{i=1}^{d} \Theta \left( \frac{p_i}{mc} \right)^2} \right] \frac{dp_1 \ldots dp_d}{\sigma^d} = \pi^{\frac{d-1}{2}} \cdot 2^{\frac{d+1}{2}} \left( \frac{\sigma}{mc} \right)^d \cdot \Theta^{\frac{d}{2}} \prod_{i=1}^{d} \Theta_i^{-\frac{1}{2}} K_{\frac{d+1}{2}} \left( \frac{1}{\Theta} \right),
\]

the mean value of a momentum function \( f(p) \) as

\[
\langle f \rangle = -\ln N + N \cdot \left( \frac{\sigma}{mc} \right)^d \cdot \frac{1}{\Theta} \cdot \sqrt{1 + \sum_{i=1}^{d} \Theta \left( \frac{p_i}{mc} \right)^2} \cdot \frac{dp_1 \ldots dp_d}{\sigma^d} = 1,
\]

and the entropy as

\[
\frac{1}{k_B} S = -\int_{-\infty}^{\infty} P_{MJ}(p; \Theta) \cdot dp_1 \ldots dp_d.
\]

The entropy is analytically derived, below. First, we derive this for the isotropic case:

\[
\frac{1}{k_B} S = -\int_{-\infty}^{\infty} P_{MJ}(p; \Theta) \cdot \ln P_{MJ}(p; \Theta) \cdot dp_1 \ldots dp_d
\]

\[
= -\ln N + N \cdot \left( \frac{\sigma}{mc} \right)^d \cdot \frac{1}{\Theta} \cdot \sqrt{1 + \sum_{i=1}^{d} \Theta \left( \frac{p_i}{mc} \right)^2} \cdot \frac{dp_1 \ldots dp_d}{\sigma^d}
\]

\[
= -\ln N + \frac{1}{\Theta} \cdot \int_{-\infty}^{\infty} P_{MJ}(p; \Theta) \cdot \gamma(p) \cdot dp_1 \ldots dp_d
\]

\[
= N(\Theta) \cdot \exp \left[ -\frac{1}{\Theta} \cdot \sqrt{1 + \sum_{i=1}^{d} \Theta \left( \frac{p_i}{mc} \right)^2} \right]
\]

\[
\times \left[ 1 + \sum_{i=1}^{d} \Theta \left( \frac{p_i}{mc} \right)^2 \right]
\]

\[
= N(\Theta) \cdot \exp \left[ -\frac{1}{\Theta} \cdot \sqrt{1 + \sum_{i=1}^{d} \left( \frac{\tilde{p}_i}{mc} \right)^2} \right]
\]

Hence, we end up with

\[
S = \frac{U}{T} - k_B \ln N = U \cdot \frac{1}{T} + k_B \ln Z.
\]

For the anisotropic case, we have

\[
\frac{1}{k_B} S = -\int_{-\infty}^{\infty} P_{MJ}(p; \Theta) \cdot \ln P_{MJ}(p; \Theta) \cdot dp_1 \ldots dp_d
\]

\[
= -\ln N(\Theta) \cdot \gamma(p, \Theta) \cdot dp_1 \ldots dp_d,
\]

because

\[
P_{MJ}(p; \Theta) \cdot \gamma(p, \Theta)
\]

\[
= N(\Theta) \cdot \exp \left[ -\frac{1}{\Theta} \cdot \sqrt{1 + \sum_{i=1}^{d} \Theta \left( \frac{p_i}{mc} \right)^2} \right]
\]

\[
\times \left[ 1 + \sum_{i=1}^{d} \Theta \left( \frac{p_i}{mc} \right)^2 \right]
\]
We now proceed to the thermodynamic relation that defines the second, the free-energy relation: 

\[ \frac{1}{k_B} S = \frac{1}{k_B} S_{\text{iso}} + \ln \left( \frac{N(\Theta)}{N(\Theta_i)} \right) \]

where \( S_{\text{iso}} \) is the entropy for the isotropic case, given by Eq. (51). In general, the product of elements of constant sum (Eq. 47) is maximized when all the elements are equal. Indeed, by maximizing the quantity \( \sum_i \ln \left( \frac{\Theta_i}{\Theta_i^0} \right) \) under the constraint of constant \( \sum_i \Theta_i^0 \) with a Lagrange coefficient \( \lambda \), we have

\[
0 = \sum_i \left[ \ln \left( \frac{\Theta_i}{\Theta_i^0} \right) + (\lambda - 1) \cdot \frac{\Theta_j}{\Theta_j^0} \right]
\]

\[
\Theta_j^{-1} - \left[ \lambda (d - 1) + 1 \right] \Theta_j^{-1} = 0
\]

\[
\Rightarrow \theta_j = \left[ \lambda (d - 1) + 1 \right] \cdot \text{constant}
\]

(Note that several thermodynamical properties can be found in Cercignani and Kremer, 2002.)

### 4.4 Thermodynamic relations

Here, we show four basic relations of thermodynamics using the isotropic theory for anisotropic temperature. Equation (51) gives the first thermodynamic relation, from which we easily derive the second, the free-energy relation:

\[ S = U/T + k_B \ln Z, \]

\[ F \equiv U - TS = -k_B T \cdot \ln Z. \]

We now proceed to the thermodynamic relation that defines temperature (Livadiotis and McComas, 2009):

\[
\frac{1}{k_B} \frac{\partial S}{\partial \Theta} = \frac{1}{\Theta^2} \frac{\partial \left( \frac{\Theta^2 U}{E_0} \right)}{\partial \Theta} - \frac{1}{\Theta^2} \frac{U}{E_0} + \frac{\partial \ln Z}{\partial \Theta}. \tag{55a}
\]

From Eq. (14c), we derive \( \frac{\partial \ln Z}{\partial \Theta} \):

\[
\frac{\partial \ln Z}{\partial \Theta} = \frac{d - 1}{2} \left( \frac{1}{\Theta^2} \right) - \frac{1}{\Theta^2} K_{\frac{d-1}{2}} \left( \frac{1}{\Theta^0} \right)
\]

\[
= \frac{d}{\Theta^2} + \frac{1}{\Theta^2} K_{\frac{d-1}{2}} \left( \frac{1}{\Theta^0} \right) = \frac{U}{\Theta^2 E_0}
\]

4.4 Thermodynamic relations

Hence, substituting Eq. (55b) into Eq. (55a), we obtain the third thermodynamic relation, the one that defines the temperature via the entropy (thermodynamic definition of temperature; Livadiotis and McComas, 2009; Livadiotis, 2015):

\[
\frac{1}{k_B} \frac{\partial S}{\partial \Theta} = \frac{1}{\Theta} - \frac{\partial \left( \frac{1}{E_0} U \right)}{\partial \Theta} = \frac{\partial S}{\partial U} = \frac{k_B}{E_0 \Theta} = \frac{1}{T}. \tag{57}
\]

This equation guarantees that the parameter \( \Theta \), selected to characterize the temperature (normalized to \( E_0 \)) in MJ distribution, was correct. Finally, from the relations \( U = U(\Theta) \) and \( \ln Z = \ln Z(\Theta) \) in Eqs. (42c) and (55b), we find the fourth thermodynamic relation:

\[
\Theta^2 \frac{\partial \ln Z}{\partial \Theta} = \frac{U}{E_0} - \frac{1}{E_0}
\]

\[
= - \frac{d+1}{2} \Theta - \left[ K_{\frac{d-1}{2}} \left( \frac{1}{\Theta^0} \right) + K_{\frac{d+1}{2}} \left( \frac{1}{\Theta^0} \right) \right] = 0
\]

\[
\Rightarrow \frac{U}{E_0} = \Theta^2 \frac{\partial \ln Z}{\partial \Theta} = - \frac{\partial \ln Z}{\partial \Theta} \quad \text{or}
\]

\[
U = - \frac{\partial \ln Z}{\partial (k_B T)^{-1}} = - \frac{\partial \ln Z}{\partial \beta_{th}} \quad \text{with} \quad \beta_{th} \equiv (k_B T)^{-1}.
\]

Note that the investigation to derive the thermodynamic definition of the temperature in the case of the anisotropic MJ, or even the classical MB, distributions is still in process.

### 5 Conclusions

The paper developed a model for the anisotropic Maxwell–Jüttner distribution, and examined its properties and thermodynamics. The Maxwell–Jüttner distribution is useful in space, geological, and other plasmas where high energy particles often reach the relativistic limits where the Maxwell–Boltzmann distribution is not applicable. On the other hand, these plasmas are known to be characterized by anisotropic Maxwell–Boltzmann distributions as long as particle energies are non-relativistic; thus, we have no reason to expect isotropic distributions at relativistic high energies. Therefore, it is necessary to deduce a consistent formulation of anisotropic Maxwell–Jüttner distributions.

First, it provided the characteristic conditions that a consistent and well-defined anisotropic model of Maxwell–Jüttner distributions needs to fulfill. Then, guided by these conditions, the paper derived a consistent model, and examined its properties and thermodynamics.
In particular, the paper showed and discussed the following analytical developments:

- provided the conditions for modeling consistent anisotropic Maxwell–Jüttner distributions,

- examined several models, showing their possible advantages and failures,

- derived a consistent anisotropic model that fulfills all the desired characteristic conditions. For example, the correspondence between both the classical and isotropic limits, for \( c \to \infty \) (anisotropic MB) and for \( T_e \to 1 \) (isotropic MJ), is fulfilled; and

- studied the properties and thermodynamics of this model, e.g., the internal energy, the partition of temperature to its components, and the entropy. The anisotropic internal energy and entropy are both dependent on the anisotropy; they are maximized for the isotropic case, while both become independent of the anisotropy at the classical case of \( c \to \infty \).

It is now straightforward for space physics researchers to use the derived analytical model in applications. The next goals may be to (i) show the connection with thermodynamics for the anisotropic model, (ii) derive other different anisotropic model(s), and (iii) to establish both the isotropic/anisotropic models within the framework of kappa distributions for particle systems described by stationary states out of thermal equilibrium (e.g., Livadiotis, 2015).

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References


